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EXACT ABELIAN HIGGS VORTICES AS 6D BRANE WORLDS

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Resumen

En este trabajo se reportan nuevas soluciones exactas a las ecuaciones del vórtice de Nielsen Olesen, para el modelo Higgs Abeliano en espacio plano en 3 dimensiones y curvo en 6 dimensiones. En el caso plano 3D, escogiendo condiciones de borde adecuadas, encontramos una equivalencia entre las ecuaciones para la cuerda vórtice y la ecuación de Schrdinger no lineal con el potencial "sombrero mexicano". También se presenta una solución exacta a las ecuaciones del vórtice con un nuevo potencial, similar al sombrero mexicano. En 6D encontramos nuevas soluciones tipo cuerda vórtice las ecuaciones de Einstein para tres potenciales diferentes todos con rompimiento espontáneo de simetría. Tal como ocurre para los mundos brana RS basados en paredes de dominio, mostramos que el modo no masivo de la gravedad linealizada está localizado en una vecindad del vórtice solución. También mostramos que no hay modos masivos ligados y encontramos las correcciones al límite newtoniano de la gravedad efectiva en 4 dimensiones.

PACS numbers: 04.20.-q, 11.27.+d, 04.50.+h

Abstract

In this work we report new exact solutions to Nielsen Olesen vortex equations for the Abelian Higgs model in flat 3 dimensional and curved 6 dimensional space times. In the 3D flat case, choosing suitable boundary conditions we found a complete equivalence between the string vortex equations for the mexican hat potential and non linear schrödinger equations. We also present an exact vortex solution with a new potential similar in shape to the mexican hat. In 6D we found new string vortex solutions to the einstein equations for 3 different symmetry breaking potentials. Just as happens for RS domain wall brane worlds, we show that the massless mode of linearized gravity is localized in the neighborhood of the vortex solution. We also show that there are not massive bounded states and found the correction to the newtonian limit of the effective gravity.

PACS numbers: 04.20.-q, 11.27.+d, 04.50.+h

1 Introduction

Recently there has been a renewed interest in the field theory on gravitational topological defects as domain walls [1], vortices, and monopoles [2]. It had been known for many years that topological defects could confine fundamental fields as matter chiral fermions [3] or gauge fields [4] when same coupling mechanism is provided; but it was only after the proposal of the brane world, in which the universe is

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considered a 3-brane contained in some large extra dimensions, with finite volume [5], that field theory on topological defects has been intensely studied.

The idea that the Universe is some kind of membrane where matter and gauge fields are excitations of the D-branes [6] contained in it, while the gravity is transmitted in the bulk between them by closed strings, was developed several years ago in the context of string theory [7]. The extra dimensions were thought to be compactified on some closed manifolds, so standard model was expected to be obtained as a very low energy limit. So far, string theory has not been able to explain fundamental problems as the cosmological constant problem and the hierarchy problem, neither the correct selection of the vacuum nor the compactification process, although many attempts had been made [8].

Brane worlds theories on domain walls had achieved successful development in recent years, specially after the important results by Randall and Sundrum explaining the hierarchy problem in a model with two branes in an AdS orbifold [9], and the confinement of spin 2 massless gravity with the classical newtonian limit [10]. Most of the development on brane world theory as confinement of chiral fermions, localization of gauge fields, BPS solutions, wide and thin domain walls limits etc. had been achieved on 5 dimensional domain walls [11]. Domain walls are not free of problems, the lost of energy by radiation, the stability under casimir energy and/or quantum fluctuation, the localization of gauge field, the supersymmetric extension and its relation with inflation and are still unsolved problems currently under research [12].

Domain walls are, by far, the most simple topological defect. The string-vortex defects are topological charged solutions to the equations of the Abelian-Higgs model, with non trivial first homotopy group for the vacuum manifold, i.e. there exist strings solutions (or holes in 2 dimensions) around which any loop will have non trivial winding number. Brane worlds had been proposed on 6 dimensional string-vortices [13] and also on 7 dimensional monopoles [14][15] on curved space time. It is natural to expect better phenomenological behavior with more extra not compactified dimensions, in the same way that smaller observable size is obtained for larger number of dimensions in AADD models [5] or greater warping volume factor arises with a greater number of warped dimensions in RS models [9][10].

Some important advances as localization of massless gravitons [16] [17], confinement of fermions [20], decoupling of graviphotons and graviscalar [18] and confinement of electromagnetic spin 1 gauge field [19][21] had been recently achieved for gravitational string-vortex in 6D, while the regularization of effective 4D theories from QED or minimal supergravity on 6D brane worlds is still under research [22]. Almost all the works relays on numerical[13][18][19] or asymptotical approximations [16][17][21][22], because there are not known general analytic solutions for curved 6D string-vortices and very few particular solutions had been reported [23], moreover in flat 2+1 dimensions only for the critical coupling exact solutions are known [24][25]. This lack of knowledge is even worst for monopoles, for which there are known exact solutions only for few special of cases [26]. Neither for string-vortex nor for monopoles there are known general solutions in curved space time, nevertheless RS scenarios based on asymptotical (and numerical) solutions had been reported [14][15][13]. In order to proceed to future developments as quantization and stability under fluctuations studies it is very desirable to obtain exact solutions. So it is the principal purpose of this paper to show new exact solutions to string-vortices, on which a RS scenario could be constructed.

In section II we review the abelian higgs vortex in flat 2+1 spacetime and show an exact equivalence between Nielsen-Olesen vortex for the mexican hat potential and non linear schrödinger equation, we also propose an exact string-vortex solution for a new symmetry breaking potential with a proper choose of the boundary conditions. In section III we obtain the exact solution of the einstein equations in 6 dimensions for 3 new symmetry breaking potentials, two of which are non polynomial, with the same kind of boundary conditions that in the former section. In section IV we give two different proofs of the localization of the zero massless mode for linearized gravity fluctuations and also prove that massive modes are not bounded to the vortex. In section V we obtain the massive exact solutions for linearized gravity fluctuations, and evaluate the newtonian limit for gravity. In the VI section we discuss same remarks and conclusions, in the appendix we give a proof that the new boundary conditions are allowed for vortex solutions.

2 An exact vortex for abelian higgs model.

First we will present an exact cylindrical solution to the Abelian Higgs model for a potential with spontaneous symmetry breaking, $V = V(\|\phi\|)$, that depends on the module of the scalar field in a general form. The Lagragean density for the Abelian Higgs model in 2+1 dimensions is

$$\mathcal{L}_{AH} = \frac{1}{2}(D_i\phi)^*(D^i\phi) - \frac{1}{4}F_{ij}F^{ij} - V, \tag{1}$$

describing a scalar field ϕ, of charge e in interaction with a Maxwell $U(1)$ field A^i and where $i, j = 0, 1, 2$ and $D_j = \partial_j - ieA_j$.

We will look for a string-vortex static solutions in cylindrical coordinates (r, θ) of the form [27][28]

$$\begin{aligned}
\phi(\overrightarrow{r}) &= f(r).e^{in\theta}, \qquad \alpha,\ f \in \mathbb{R}, \quad n \in \mathbb{Z} \\
A_a(\overrightarrow{r}) &= -\varepsilon_{ab}\,\frac{n}{e}\frac{x_b}{r^2}\alpha(r), \qquad a, b = 1, 2 \\
A_o(\overrightarrow{r}) &= 0.
\end{aligned} \tag{2}$$

With the replacement of (2) into Euler Lagrange equations for (1) we get the Nielsen-Olesen vortex equations:

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2 f}{r^2}(\alpha - 1) - \frac{dV}{df} = 0, \tag{3}$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} - e^2 f^2(\alpha - 1) = 0. \tag{4}$$

String-vortex solutions were obtained with the boundary conditions:

$$\begin{aligned}
f(r \to 0) = 0, \quad f(r \to \infty) = 1, \\
\alpha(r \to 0) = 0, \quad \alpha(r \to \infty) = 1,
\end{aligned} \tag{5}$$

by numerics and asymptotical methods. Exact solutions to the equations (3,4) are not known except for the mexican hat potential in the Bogomol'nyi case, in which case, a critical relation between the coupling constant and the electric charge is required [24].

We will relax the boundary conditions taking $\alpha(r \to 0)$ finite instead of $\alpha(r \to 0) = 0$, so we will use:

$$\begin{aligned}
f(r \to r_c) = 0, \qquad f(r \to \infty) = 1, \\
\alpha(r \to r_c) = \text{constant}, \quad \alpha(r \to \infty) = 1,
\end{aligned} \tag{6}$$

These new boundary conditions (6) are allowed although $\alpha(r \to 0)$ finite could lead to a singularity in (2) in the gauge field $A_a(\overrightarrow{r})$. There is no problem with single pole singularities at the gauge connection. The obvious example is the static electrical monopole (free point charge) its conecction $A_o(r) = \frac{e}{r}$ is singular at the charge $r = 0$. The same happens with the Dirac´s Monopole, that indeed have a singularity when $r \to 0$. This could be easily seen in Wu and Yang potential for the monopole with two local charts: $A^{+,-} = \pm\frac{g(1\mp\cos\theta)}{r\sin\theta}\hat{u}_\phi$ for the northern H^+ and southern H^- hemispheres [38]:

Boundary conditions (6) together with equation (4) suggest the following ansatz that will be proved in the appendix:

$$\alpha(r) \equiv 1, \qquad \text{for all } r \tag{7}$$

gauge invariant conditions as the magnetic flux, in this case, is easily obtained from (2) and (7) for any closed curve C containing the origin

$$\frac{2\pi n}{e} = \int_C A \cdot dl,$$

so we have a $\Pi_1 = \mathbb{Z}$ vortex solution. Equation (4) is automatically satisfied, and equation (3) takes the form:

$$\nabla^2 f = \omega(f), \qquad \text{where} \qquad \omega(f) = \frac{dV}{df}. \tag{8}$$

For the mexican hat potential

$$V_m = \frac{\lambda_m}{4}(f^2 - \eta^2)^2, \qquad \text{where} \qquad \eta = cte, \tag{9}$$

equation (8) takes the form of a non linear schrödinger

$$-\nabla^2 f + \lambda_m \|\phi\|^2 f = \lambda_m \eta^2 f. \tag{10}$$

Non linear quantum mechanical potential could be thought as $V_{NLQ} = \lambda_m \|\phi\|^2$, and as $f(r \to \infty) = 1$ by (6), it is necessary that $\eta < 1$ in order to allow the existence bounded states with $\lambda_m \eta^2$ ëigenvalue [29].

So we have found that non linear Schrödinger solutions are a complete set of solutions to Nielsen Olesen equations (3,4) for the abelian higgs model (1) when the mexican hat potential (9) is used.

Now we will look for an exact solution to (3,4) with the following potential:

$$\begin{aligned} V(\|\phi\|) &= \frac{\lambda}{2}(\|\phi\| - 1)^2, \\ V(f) &= \frac{\lambda}{2}(|f| - 1)^2, \end{aligned} \tag{11}$$

this term could be interpreted as a mass term in (1) by redefining $f \to f+1$ in (11), but we rather prefer to interpret it as a symmetry breaking potential similar to (8), in order to obtain exact solutions by the simplification of the equation (3):

$$\frac{d^2 f^+}{dr^2} + \frac{1}{r}\frac{df^+}{dr} = \lambda(f^+ - 1) \tag{12}$$

$$\frac{d^2 f^-}{dr^2} + \frac{1}{r}\frac{df^-}{dr} = \lambda(f^- + 1) \tag{13}$$

where f^+ stands for $f(r)$ when $f \geq 0$, and f^- stands for $f(r)$ when $f < 0$.

Equation (12) for $f \geq 0$ has simple exact solutions given by

$$f^+ = 1 + a^+ J_0(i\sqrt{\lambda}r) + b^+ Y_0(-i\sqrt{\lambda}r), \qquad a^+, b^+ \in \mathbb{C} \tag{14}$$

where J_0 and Y_0 stands for the first and second kind bessel function of zero order.

As $J_0(i\sqrt{\lambda}r)$ and $Im[Y_0(-i\sqrt{\lambda}r)]$ are divergent when $r \to \infty$, we choose:

$$a^+ = i.c, \quad b^+ = c, \tag{15}$$

in order to satisfy the boundary conditions (6) and using the definitions of K_0 second kind bessel function we get

$$f^+_{out} = 1 - \frac{2c}{\pi} K_0(\sqrt{\lambda}r), \tag{16}$$

which is valid for $r > r_c$ where r_c is such that $f(r_c) = 0$,for $r < r_c$ that point f becomes negative ($f(r) < 0$), so equation (12) is not longer acceptable. The radius r_c can be interpreted as the "core radius" of the string-vortex, outside which (16) must be seen as the exterior solution to vortex equation (3).

In a similar way for $f < 0$ we found the solution to (13)

$$f^- = -1 + a^- J_0(i\sqrt{\lambda}r) + b^- Y_0(-i\sqrt{\lambda}r), \qquad a^-, b^- \in \mathbb{C}$$

in order to give a real non divergent solution, we take

$$a^- = -i.c, \quad b^- = -c,$$

for which the solution results in

$$f^-_{out} = -[1 - \frac{2c}{\pi} K_0(\sqrt{\lambda}r)], \tag{17}$$

that is the solution (16) with reverse sign, and it is also valid for $r > r_c$. i.e. outside the string core, so it is physically the same solution than (16) with a unphysical phase change by π.

It is remarkable that this vortex solution (16) for the potential (11), is exactly the same than asymptotically solution [27] known very long ago for the mexican hat potential. But in this case, the solution (16) with boundary conditions (7) and (6) is exact. As this new string-vortex solution is the same that the known asymptotically solution (2), it has the same integer winding number and positive definite energy rendering a physically acceptable solution.

The interior solution ($r < r_c$) also can be obtained by setting

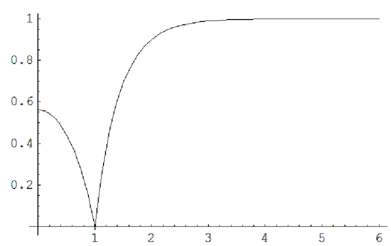
$$a^+ = -k, \quad b^+ = 0,$$
$$a^- = k, \quad b^- = 0,$$

so:

$$f^{\pm}_{in} = \pm[1 - k.J_0(i\sqrt{\lambda}r)], \tag{18}$$

that indeed is real and finite for $r < r_c$.

A complete solution to (12) can be constructed for all r using (16,17) and (18) as it shown in Fig.1.



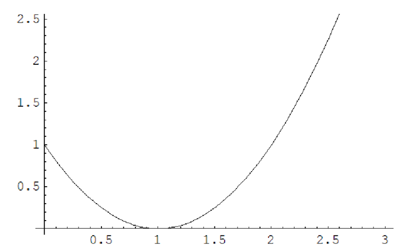


Figure 1: Plots of the functions f^+_{in} (k=0.44, $\lambda = 4, r < 1$) and f^+_{out} (c=4.4, $\lambda = 4, r > 1$)(left) and the symmetry breaking potential eq. (11) (right).

3 New exact einstein abelian string-vortex solutions.

We start from the 6 dimensional action

$$S = -\int dx^6 \sqrt{-G}\left[\frac{1}{2\chi}R + \Lambda\right] + \int dx^6 \sqrt{-G}\left[\frac{1}{2}(D_A\phi)^*(D^A\phi) - \frac{1}{4}F_{AB}F^{AB} - \widetilde{V}\right], \tag{19}$$

where the first integral is the 6D Einstein Hilbert action, with a bulk cosmological constant Λ.In what follows we will set $\chi = \frac{8\pi}{(M_6)^4} = 1$, where M_6 is the 6D Planck mass and we will follow the notation in [13][1].

[1] A, B and uppercase Latin indices run over 4+2 dim. μ, ν and Greek indices over 4 dim. middle alphabet Latin indices run in 3 dim $i, j = 0, r, \theta$ and firsts alphabet Latin indices run over 2 dim $a, b = r, \theta$.

Here we will look for a geometry 3+2+1 composed by a 3-brane that contains the 3+1 physical universe and 2 extra dimensions, where we can choose coordinates (r, θ) as in the former section and the metric is given by:

$$ds^2 = M^2(r) g_{\mu\nu} dx^\mu dx^\nu - L^2(r) d\theta^2 - dr^2, \tag{20}$$

with $g_{\mu\nu} = \eta_{\mu\nu} = (+, -, -, -)$. In this context r and θ are the coordinates of the extra dimension, $L(r)$ acts as a radial factor and $M(r)$ as warp factor for a 4 dimensional brane when move into the extra dimensions.

Assuming that the scalar and gauge fields in (19) depends only on the extra coordinates and using the Nielsen Olesen ansatz (2) and setting $A_\mu = 0$ for all brane world 4D coordinates, we get the curved version of (3,4):

$$\frac{d^2 f}{dr^2} + (4m + l)\frac{df}{dr} - \frac{P^2 f}{L^2} - \frac{dV}{df} = 0, \tag{21}$$

$$\frac{d^2 P}{dr^2} - (4m + l)\frac{dP}{dr} - e^2 f^2 P = 0, \tag{22}$$

where

$$P = n\,[1 - \alpha(r)]; \qquad m = \frac{d}{dr}\ln[M(r)]; \qquad l = \frac{d}{dr}\ln[L(r)].$$

As ansatz (7) was useful to find exact solutions in the flat spacetime case, we will assume here:

$$P \equiv 0, \tag{23}$$

so the maxwell equation (22) is automatically satisfied.

Einstein equations are obtained performing variations of the metric in the action (19) as in [13] and [17].

$$l' + 3m' + l^2 + 6m^2 + 3ml = -\tau_o, \tag{24}$$

$$4m' + 10m^2 = -\tau_\theta, \tag{25}$$

$$2ml + 3m^2 = -\frac{1}{2}\tau_r, \tag{26}$$

where $'$stands for $\frac{d}{dr}$, and τ_o, τ_θ, τ_r are the non vanishing components of the energy momentum tensor:

$$\begin{aligned}
\tau_o &= T^0_0 = \frac{(f')^2}{2} + V(f), \\
\tau_\theta &= T^\theta_\theta = \frac{(f')^2}{2} + V(f), \\
\tau_r &= T^r_r = -\frac{(f')^2}{2} + V(f),
\end{aligned} \tag{27}$$

here (23) was inserted into the equations and the 6D (bulk) cosmological constant was absorbed into the redefinition of the potential $V = \tilde{V} + \Lambda$.

The equation for the scalar field (21) could also be obtained, as in [15][17], as a consequence of the conservation of energy momentum tensor, by means of:

$$\frac{d}{dr}\tau_r = 4m(\tau_o - \tau_r) + l(\tau_\theta - \tau_r), \tag{28}$$

so, given a potential $V(f)$, equation (21) is functionally dependent of the system (24,25,26,28).

We will consider the potential $V(f)$ also as a variable of the system of differential equations (24,25,26) jointly with (21), That is a system of 4 equations with 4 variables (m, l, f, V).With straightforward combinations, the complete system could be written as:

$$f'' + (4m + l)f' - \frac{d}{df}V = 0, \quad (29)$$

$$l' + (4m + l)l = -\frac{1}{2}V, \quad (30)$$

$$m' + (4m + l)m = -\frac{1}{2}V, \quad (31)$$

$$m' + m^2 - ml = -\frac{1}{4}(f')^2. \quad (32)$$

Equations (30) and (31) implies

$$m = l, \quad (33)$$

that can be obtained exactly considering solutions by separation of variables: $l = q(r)\ m$, because the only consistent function with the system is $q(r) = 1$.

So the system reduces to:

$$f'' + 5m.f' - \frac{d}{df}V = 0, \quad (34)$$

$$m' + 5m^2 = -\frac{1}{2}V, \quad (35)$$

$$m' = -\frac{1}{4}(f')^2. \quad (36)$$

as before for a given $V(f)$ the system (34,35,36) is not longer independent, and always one of the equations can be obtained from the other two. But still solving the system is not easy, due to its coupled non linear nature.

We will follow here the approach developed in [30] and [31]. Instead of try to solve for a given $V(f)$, we will give a probe function $f(r)$ that accomplishes the boundary conditions (6), from (36) it is easy to obtain m as:

$$m = -\frac{1}{4}\int dr(f')^2 - k_{RS}, \quad (37)$$

where k_{RS} is an additional Randall Sundrum constant warp factor. Then from equation (35) we could obtain the potential $V(r)$ as a function of r. In order to obtain the interaction potential $V(f)$ we must solve $r(f)$ from probe function $f(r)$ so:

$$V(f) = V(r(f)), \quad (38)$$

The potential V(r) could also be obtained from equation (34), but from (35) is easier. Equation (28) is guaranty that (34) is accomplished. Of course a solution to be physically acceptable must have a stable potential with spontaneous breaking of symmetry, as we will show immediately.

Equations (34,35,36) are very similar to that of 5D domain walls in [30] and 4 dimensional domain walls in [32], so we will try:

$$f = f_0 \arctan\left(\sinh\ \beta r/\delta\right), \qquad f_0 = 2\sqrt{\delta}, \quad (39)$$

using (37) we get

$$m = -\beta \tanh\left(\beta r/\delta\right), \quad \text{where we set} \quad k_{RS} = 0, \quad (40)$$

That correspond to the metric "warped" factor

$$M(r) = \cosh^{-\delta}(\beta r/\delta), \qquad \delta > 0, \qquad \beta > 0. \quad (41)$$

From (35) we obtain the potential $V = V(r)$ as:

$$V(r) = \frac{2\beta^2}{\delta}[(1+5\delta)\sec h^2(\beta r/\delta) - 5\delta], \tag{42}$$

and solving r as a function of f by means of (39) we have

$$\cos^2(\frac{f}{f_o}) = \sec h^2(\frac{\beta r}{\delta}), \tag{43}$$

Finally the potential is

$$V(f) = \frac{2\beta^2}{\delta}[(1+5\delta)\cos^2(f/f_o) - 5\delta]. \tag{44}$$

That is an stable potential, with spontaneous symmetry breaking minima, where the scalar field interpolates smoothly between $f = \pm\pi\sqrt{\delta}$ in AdS spacetime with cosmological constant $\Lambda = 10\beta^2$ as could be seen from and Fig.2.

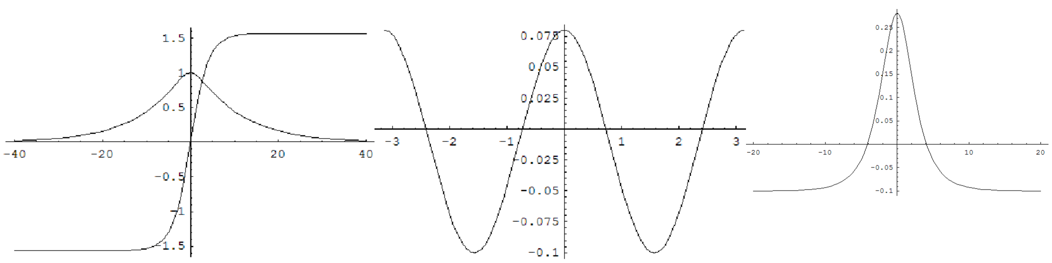


Figure 2: Graphic of the metric factor M and scalar field (left), the potential (center), and the energy density (right). fo=1, $\delta = 1/4$, $\beta = 0.1$

Here we will present another two new solutions, these are found by setting:

$$\begin{aligned} f &= f_0 \tanh(a\ r), \qquad \text{or} && (45)\\ f &= f_0 \arctan(a\ r), && (46)\end{aligned}$$

using (37) we get $m(r)$, and setting $k_{RS} = 0$ as before, we obtain the metric "warped" factors, respectively

$$\begin{aligned} M(r) &= \exp[\frac{1}{24}f_0^2\ \sec h^2(a\ r)]\cosh^{-f_o^2/6}(a\ r), \qquad \text{and} && (47)\\ M(r) &= \exp[-\frac{a}{2}f_0^2\ r\arctan(a\ r)], && (48)\end{aligned}$$

From (35) we obtain the potential $V = V(r)$ and solving r as a function of f as before we get the potentials, respectively

$$\begin{aligned} V(f) &= \frac{a^2}{2}[(f^2 - f_o^2)^2 - \frac{5}{9}f^2(f^2 - 3f_o^2)], \qquad \text{and} && (49)\\ V(f) &= \frac{a^2 f_o^2}{2}[\cos^4(f/f_o) - \frac{5}{16}f_o^2[f/f_0 + \frac{1}{2}\sin(f/f_o)]^2 && (50)\end{aligned}$$

The potentials (49) and (50) with the metrics warped factors (47) and (48) that correspond to the scalar "kink" fields (45) and (46) are very similar in shape to the former solution (44),(41) and (39) plotted in Fig.2. As before the potentials presents symmetry breaking minima, and the scalar field is also an interpolating soliton between those minima in AdS space time with cosmological constant $\Lambda = -\frac{5}{18}a^2 f_0^4$ and $\Lambda = -\frac{5}{128}a^2\pi^2 f_0^4$, respectively.

Remarkably these string-vortex solutions are exact. Until now almost all solutions found, in 6D curved space-time, were asymptotical or numerical [13][17] with the exception of [23]. If we set $f = 1$ in (34,35,36) we obtain $M = \exp[-\beta r]$ in concordance with [23]; but in that case the potential is constant and do not show breaking of symmetry. In general this kind of solution are very important to establish the possibility of confinement, quantization and stability of other fields as fermions, photons, gravitons etc. and could make possible the construction of a brane world on that background opening the possibility to futher developments.

4 Localized gravity on the vortex

Next, we will show that these new vortices can confine gravity. In the previous section we found an exact solution to the einstein equations for the metric (20,33,41) for the case that the gauge field and scalar field are given by (2,23,39) with the non trivial potential (44). We will study linearized spin 2 metric fluctuations from the metric (41) neglecting graviscalar and graviphotons modes.

$$h_{\tau\nu}(x,r,\theta) = \eta_{\tau\nu} e^{ip\cdot x} \sum_{\kappa,\mu} \phi_\mu(r) e^{i\kappa\theta}, \tag{51}$$

where x stands for x^μ the cordinates on the 4 dimensional branes and (r,θ) for the polar coordinates of the bulk. These massive modes (51) obeys the Laplace Beltrami operator, and straightforward the radial modes satisfy [33]

$$-\frac{1}{M^2 L}\frac{\partial}{\partial r}\left[\sqrt{-g}\frac{\partial}{\partial r}(\phi_\mu)\right] = \mu^2 \phi_\mu, \tag{52}$$

where $\mu^2 = (m_o)^2 - \left(\frac{M}{L}\right)^2 (\kappa)^2$ represents the mass of mode ϕ_μ with 4 dimensional momentum $p_\nu p^\nu = m_o^2$. The mass term takes into account the contributions from orbital angular momentum κ, as in [17].

Upon substitution of (33,41) we get the equation for massive metric fluctuations:

$$-\phi''_\mu + 5\beta \tanh(\beta r/\delta)\phi'_\mu - \mu^2 \cosh(\beta r/\delta)\phi_\mu = 0, \tag{53}$$

when $\mu^2 = 0$ the equation (53) reduces to:

$$\phi'_o \cosh^{-5\delta}(\beta r/\delta) = k_1, \tag{54}$$

so integrating (54), we obtain the massless mode

$$\phi_o = k_o + k_1 \int dr \cosh^{5\delta}(\beta r/\delta). \tag{55}$$

with integrations constants k_o and k_1.

As $\cosh^{5\delta}[\beta r/\delta]$ is monotonous growing, we must fix $k_1 = 0$, in order to render ϕ_o normalizable. That is consistent with the boundary conditions

$$\phi'_\mu(0) = \phi'_\mu(\infty) = 0, \tag{56}$$

that allows (52) to be self adjoint [17].

The ortonormalization condition to be satisfied by ϕ_μ is

$$\begin{aligned} \int_0^\infty dr\ M^2 L\ \phi^*_\mu(r)\phi_\upsilon(r) &= \delta_{\mu\upsilon}, \\ \int_0^\infty dr \cosh^{-3\delta}(\beta r/\delta)\ \phi^*_\mu . \phi_\upsilon &= \delta_{\mu\upsilon}, \end{aligned} \tag{57}$$

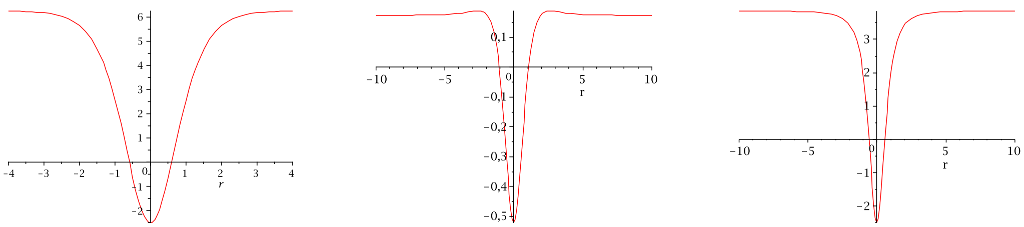


Figure 3: Graphics of the potential V_{QM} for the massless mode of the vortices with M given by (41) left ($\beta = 1,\ \delta = 1$); with (47) center and (48) right (fo=1, a=1).

so the equivalent wavefunction in 1dimensional quantum mechanics is

$$\psi_\mu(r) = \cosh^{-3\delta/2}(\beta r/\delta)\ \phi_\mu(r). \tag{58}$$

Finally, the massless normalized equivalent wavefunction is given by:

$$\psi_o(r) = k_o \cosh^{-3\delta/2}(\beta r/\delta). \tag{59}$$

This function is strongly decaying (see Fig.5), so we conclude that the massless, spin two, gravitation mode is localized on the 3-brane and strongly concentrated around $r = 0$ as we expected for a RS scenario [13][17].

There is another approach to study the localization of gravity modes around the vortex [34][35] in an exact way, the equation (53) could be written as

$$-\phi_\mu'' - 5\frac{M'}{M}\phi_\mu' = \frac{\mu^2}{M^2}\phi_\mu, \tag{60}$$

that could be converted with the change of variables

$$u_\mu(r) = M^{5/2}\phi_\mu(r), \tag{61}$$

into a Schrödinger eingënvalue equation with zero eingënvalue energy

$$\left(-\partial_r^2 + V_{QM}\right)\ u_\mu = 0, \tag{62}$$

where

$$V_{QM} = \frac{15}{4}\left(\frac{M'}{M}\right)^2 + \frac{5}{2}\left(\frac{M''}{M}\right) - \left(\frac{\mu}{M}\right)^2. \tag{63}$$

For the case of solution(41), the quantum potential for massive states is:

$$V_{QM} = -\mu^2\cosh^{2\delta}(\beta r/\delta) + \frac{5\beta^2}{2}\left[\frac{5}{2}\tanh^2(\beta r/\delta) - \frac{1}{\delta}\sec h^2(\beta r/\delta)\right]. \tag{64}$$

Due to the first term in the quantum potential (64) there will be not bounded states except for $\mu = 0$, as could be seen from Fig.3. When $\mu \neq 0$, the potential is not bounded from below, as could be observed from Fig.4 so the potential well is unstable and the massive states will easily tunnel outside the well. Therefore only the massless state will be bounded.

In a similar way the equivalent quantum mechanical potential for the other two vortex solutions (45,47) and (46,48)could easily be obtained from equation (63), we can again observe in Fig.3 that when $\mu = 0$, there exist a potential well that allows the existence of a confined massless gravity ground state, but when $\mu \neq 0$, the potential has a negative exponential growing, as could be seen in Fig.4, so is not bounded from below and massive bounded states are forbidden.

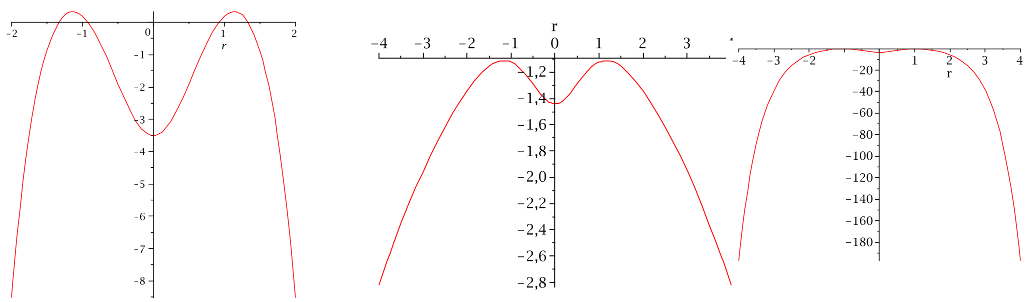


Figure 4: Graphics of the potential V_{QM} for the massive modes $\mu = 1$ of the vortices with M given by (41) left ($\beta = 1$, $\delta = 1$); with (47) center and (48) right (fo=1, a=1).

5 Massive gravity modes and newtonian limit of gravity.

In order to compute the newtonian limit of gravity we need exact solutions to massive modes equation (53). This equation can be solved for particular values of δ in term of Heun confluent H_C functions [36]:

For $\delta = 1$ we get for $\phi_\mu(r)$:

$$C_1\ H_C[0, -\frac{1}{2}, -3, -\frac{\mu^2}{4\beta^2}, \frac{4\beta^2+\mu^2}{4\beta^2}, -\sinh^2(\beta r)] + C_2\ \sinh(\beta r)\ H_C[0, \frac{1}{2}, -3, -\frac{\mu^2}{4\beta^2}, \frac{4\beta^2+\mu^2}{4\beta^2}, -\sinh^2(\beta r)], \tag{65}$$

while for $\delta = 2$ we get for $\phi_\mu(r)$:

$$C_1\ e^{\frac{-i\mu}{\beta}\sinh^2(\frac{\beta r}{2})}\ H_C[\frac{2i\mu}{\beta}, -\frac{1}{2}, -\frac{11}{2}, -\frac{\mu^2}{\beta^2}, \frac{13\beta^2+8\mu^2}{8\beta^2}, -\sinh^2(\frac{\beta r}{2})] + \tag{66}$$
$$C_2 \sinh(\frac{\beta r}{2})\ e^{\frac{-i\mu}{\beta}\sinh^2(\frac{\beta r}{2})}\ H_C[\frac{2i\mu}{\beta}, \frac{1}{2}, -\frac{11}{2}, -\frac{\mu^2}{\beta^2}, \frac{13\beta^2+8\mu^2}{8\beta^2}, -\sinh^2(\frac{\beta r}{2})].$$

A plot of the massive wavefunction (58) is given when $\delta = 1$, $\mu = 1$ and $\beta = 1$ using (65) is given in Fig. (5), where its oscillating growing nature could be observed.

Although a general solution is not known, what is really important is the asymptotic behavior of the massive modes. Far from the vortex core, i.e. $r >> r_c$ see (16), or equivalently for $\delta \to 0$, that correspond to the thin domain wall analogue, we can approximate (41) by

$$M = \cosh^{-\delta}(\beta r/\delta) \cong (\frac{1}{2})^{\delta}.e^{-\beta r}, \tag{67}$$

for the other two vortices (45,47) and (46,48), we were not able to found exact solutions to equation (52), but we can obtain the same asymptotic limit (67), with $\delta = f_o^2/6$ and $\beta = a\ f_o^2/6$ for the second exact solution (47) and $\delta = 0$ and $\beta = a\pi\ f_o^2/4$ for the third exact solution (48). So the massive wavefunction (53) could be approximate for all these three vortex solutions by

$$\psi_\mu(r) \cong e^{-\frac{3}{2}\beta r}\ \phi_\mu(r), \tag{68}$$

and the localized zero mode by

$$\psi_o(r) \cong \sqrt{3\beta}e^{-\frac{3}{2}\beta r}\ , \tag{69}$$

The massless zero mode is then localized in the vicinity of r=0, that is near 3 brane where the known universe is located, and decaying exponentially when when r is increased. The Fig.5 show the behavior of the massive and massless wave functions.

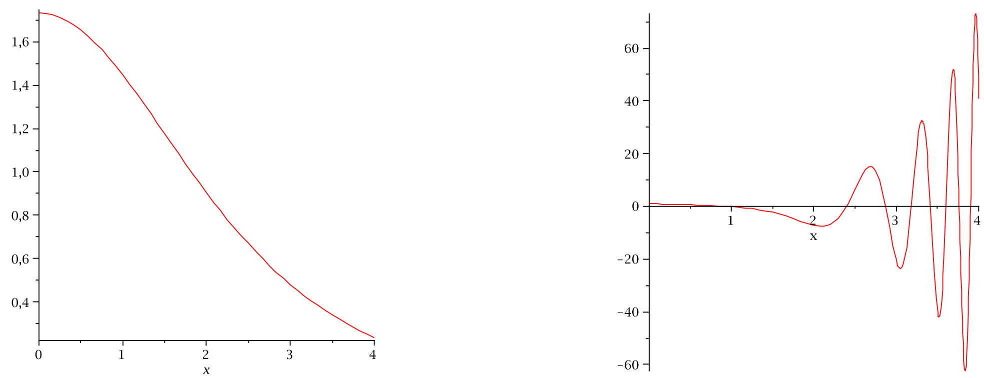


Figure 5: Plots of the massless wavefunction (left) $\mu = 0$, $\beta = 1$ Eq.(59) and massive (right) wavefunctions $\mu = 1$, $\beta = 1$ Eq.(65,58)

Using the approximation (67) and (68) into the differential equation (53), and taking the limit $\beta r/\delta \to \infty$ for which $\tanh(\beta r/\delta) \to 1$, we obtain

$$-\phi''_\mu + 5\beta.\phi'_\mu - \mu^2 e^{2\beta r}\phi_\mu = 0,$$

which solution is given in term of bessel functions $J_{5/2}(\frac{\mu}{\beta}e^{\beta r})$ and $Y_{5/2}(\frac{\mu}{\beta}e^{\beta r})$. In exact concordance with the result in [17].

The newtonian limit was obtained in [17], using a radial cutoff approach calculation as:

$$V \cong \frac{1}{m_{pl}^2}\frac{m_1.m_2}{R}\left[1 + \frac{4}{3\pi\beta^3}\frac{1}{R^3} + ...\right], \tag{70}$$

where $m_{pl}^2 = \frac{2\pi}{3\beta}(M_6)^4$ is the 4 dimensional Planck mass, and R is the distance between two particles of masses m_1 and m_2 on the 3-brane. Therefore the $\frac{1}{R^3}$ correction to the newtonian gravity is stronger than $\frac{1}{R^2}$ correction in 5 dimensional RS domain walls [10] due to the additional dimension.

6 Summary and outlook

In this work we present a proof of the equivalence between the classical Nielsen Olesen vortices and the non linear schrödinger solitonic solutions for the old abelian higgs model with the mexican hat potential. We also found a new vortex solution for the abelian-higgs model in 2+1 flat spacetime, this solution was found for a polynomial potential similar to the classical mexican hat, with similar breaking symmetry minima degeneracy. Although the solutions were found for a set of boundary conditions (6) different to the classical one (5), the radial function f^+_{out}, equation (16), has exactly the same asymptotical and near core behaviour of classical Nielsen-Olesen [27] and Bogomol'nyi [24] known solutions, as could be observed from Fig.1. We also proposed a solution inside the core f^+_{in} equation (18), that could be useful for studies of superconductivity interface analogs of classical abelian-higgs vortices.

We consider another central result of this paper, is to have found exact string-vortex brane world solutions embedded in curved 6D spacetime. These solutions, were obtained with the same kind of boundary conditions that in the 2+1 flat case, following the procedure developed in [30] and [31] to obtain solutions to einstein equations with a proposed scalar kink source while letting free the potential.

We found three new exact 6D vortex solutions, for three different kinks, that renders adequate warped metric factors and potentials with degenerate minima that allows spontaneous symmetry breaking of the theory. The first solution (39), (41) and (44) is identical to the domain wall solution in [32] where δ could be seen as the wall thickness and β as a parameter depending on the cosmological constant. The second solution (45), (47) and (49) has a polynomial potential that is almost identical to the mexican hat potential. The third solution (46), (48) and (50) combine polynomial and not polynomial terms in the potential. All the new solutions presented here, are very similar to the domain walls, but the new solutions incorporates axial symmetry in 6 dimension and a gauge background with integer winding number, that is absent in 5 dimensional domain walls.

As expected for a RS brane world we have shown that the zero mode (55) of the linearized gravity spectrum is localized on the 3-brane and that the massive modes (58,65) are not bounded at all, these results are obtained in an exact form for the 6D vortex solutions we found. Also the newtonian limit is obtained in concordance with [17].

Using exact solutions many interesting properties of 6D brane world could be study, if we set the classical solutions we have found as background, we can proceed to study the quantization and stability under fluctuations, also we can study the confinement of electromagnetic spin 1 and other gauge or fermionic fields in an exact way. These subjects are currently under research [37].

Acknowledgments

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Appendix

Following Wu and Yang classical explanation of the monopole it can be proved that boundary conditions (6) and equation (7) are allowed. Take a circle of radius $r = r_c$ on the plane $\mathbb{R}^2$ dividing it into: D^- a disk with a hole at $r = 0$,where the singular solution (2,7,18) indeed has a single pole, and D^+ is the outside of the disk $r \geq r_c$ where the solution (2,7,17) holds. Now take an sphere of radius r_c so that the plane $\mathbb{R}^2$ divides the sphere into two hemispheres H^+ for the one containing the North Pole and H^- for the one containing the South Pole, the equator will correspond to the $r = r_c$ circle on the plane.

As usual, by stereographic projection H^+ can be projected into D^+ with the exception of the North

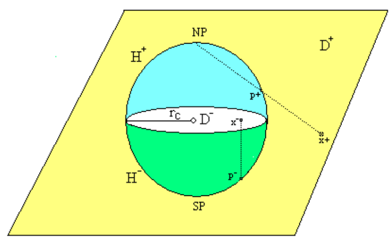

Figure 6:

Pole (NP). So D^+ is topological equivalent to $H^+ - \{NP\}$, that is a hemisphere with a hole. Simply by vertical projection (orthogonal to the plane) the hemisphere H^- can be projected into D^-, with the exception of the South Pole (SP) that is projected in the hole $r = 0$. So D^- is topological equivalent to $H^- - \{SP\}$, that is another hemisphere with a hole.

As these two hemispheres are joined by the equator $r = r_c$ this topology is equivalent to a sphere with two holes (both north and south poles removed), that is just a cylinder with open ends. It is already known that this topology has integer first homotopy group. That means that any closed curve in the plane with $r = 0$ inside, that wraps n times around the origin it will be map into a curve that circles around the cylinder n times, so the winding number is n and the first homotopy group is $\Pi_1 = \mathbb{Z}$ for this vortex.

The transition function is the difference between the connections $A(r)$ defined on the intersection $D^+ \cap D^-$, that is simply the equator. from (2) and (7) we obtain $A^+(r_c) - A^+(r_c) = 0$, so the connection (2,7) is trivially is a well defined gauge function, while the scalar field is also continuous due to $\phi^+(r_c) = \phi^-(r_c) = 0$, so it is consistent. The value of the radius r_c is dependent on the boundary condition (6) and (18) determines the value of f at the origin. Reciprocally for a given value of $f(r = 0)$, the radius r_c could be obtained solving (18).

Although the solution (2,7,18) takes singular values at $r = 0$, this point was excluded, so (6) and (7) are perfectly allowed in the Topology $\{D^+, D^-\}$ describing an electromagnetic vortex with $\Pi_1 = \mathbb{Z}$.